A genetic method for non-associative algebras (Theory of Biomathematics and its Applications VII)

AUTHOR(S):

Micali, Altibano; Suzuki, Osamu

数理解析研究所講究録
第 1751 巻 2011 年 9-17

A genetic method for non-associative algebras

By

Altibano Micali* and Osamu Suzuki **

* *Department of Mathematical Sciences , University of Montpellier II, Place Eugene Bataillon, Monytpellier, France*

** *Department of Computer and System Analysis College of Humanities and Sciences, Nihon University156 Setagaya,Tokyo,Japan*

E-mail: (A.Micali)micali@math.univ-monttp2.fr(O.Suzuki)osuzuki@cssa.chs.nihon-u.ac.jp

Keywords:Mendel's law, Jordan algebra and Context free language

Abstract. *A genetic method for a systematic construction of non-associative algebras is presented. A concept of Mendel algebra is introduced and it is proved that a certain class of Jordan algebras and flexible algebras can be found in Mendel algebras and that a classification theory of non-associative algebras based on the Mendel algebras is given from a point of view in genetics.*

Introduction

In this paper we introduce a method of genetics to non-associative algebras and generate them by use of the mathematical formulations of Mendel's law systematically and classify them based on these laws. By these observations we can conclude that the theory of genetics will be important for the theory of non-associative algebras.

We introduce a concept of Mendel algebras following the Mendel's separation law in genetics. We call the linear space M with generators $S_1, S_2, \ldots, S_n$ Mendel algebra, when generators satisfy the following commutation relations and the distributive law:

$$S_i * S_j = \frac{1}{2}\{S_i + S_j\}$$

At first we notice that the algebra is non-associative. We want to find non-associative algebras including the flexible algebras and Jordan algebras in Mendel algebras. These algebras satisfy the following commutation relations: For any pair of elements $^{\forall}X, ^{\forall}Y$ of the algebra we have the following relations respectively:

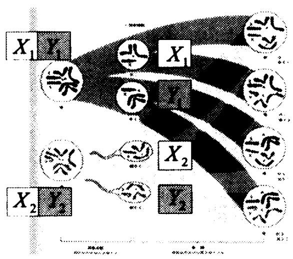
flexible algebra: $(XY)X = X(YX)$
Jordan algebra: $(((XX)Y)X) = (XX)(YX)$

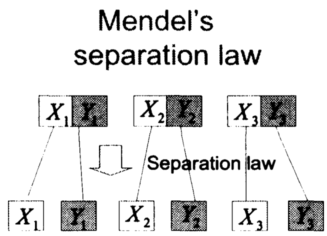
The main results of this paper can be stated as follows:
(1)Mendel algebra is flexible algebra and Jordan algebra(Theorem I and II).
(2)A family of flexible algebras and Jordan algeberas can be generated by mathematical formulation of Mendel's laws:Separation law, mating law and independent law(Theorem III).
(3) We can give a classification of non-associative algebras by use of the shift invariance condition in Mendel algebras. We can discuss these commutation relations in terms of „shift invariant elements" of an algebra. Then we can show that the shift invariant algebras on Mendel algebra automatically derive a family of non-associative algebras including flexible algebras and Jordan algebras

1. Mendel's laws

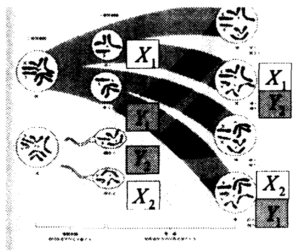
In this section we recall some basic facts on Mendel's law([4]). In 1860, Mendel has discovered the fundamental laws in genetics, which are called Mendel's laws. They constitute three laws: (1)Separation law, (2) Mating law(3) Independent law: We describe the laws with figures and we omit its description.

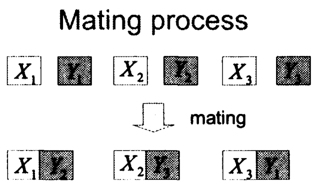
(1) Separation law





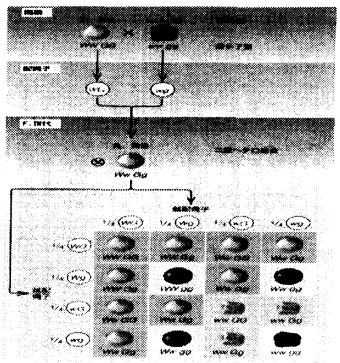
(2) Mating law

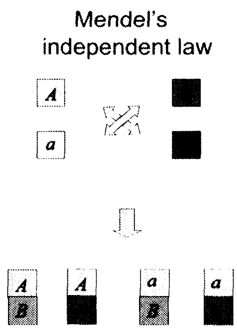




(Each element is commutative)

(3) Independent law





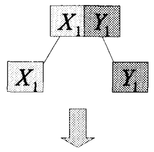
2. Mendel algebra and its basic facts

In this section we introduce a concept of Mendel algebra which is motivated by the Mendel's separation law([5]):

(Mendel algebra)

Definition

Let $A(=R[S_1,S_2,...,S_n])$ be an algebra. Introducing the product by



$X_1 * Y_1 = \frac{1}{2}(X_1+Y_1)$

$$\begin{cases} S_i * S_j = \frac{1}{2}\{S_i + S_j\} \\ X * Y = \sum_{i,j=1}^{n} \alpha_i \beta_j S_i * S_j (X = \sum_{i=1}^{n} \alpha_i S_i, Y = \sum_{i=1}^{n} \beta_i S_i) \end{cases}$$

we have an algebra $M^{(n)}(R)$ which is called n-dimensional Mendel algebra .

We notice that the Mendel algebra is non-associative and commutative algebra. In fact we can give a simple example:

$$((S_i * S_j) * S_k) = \frac{1}{4}(S_i + S_i + 2S_k), \quad ((S_i * (S_j * S_k)) = \frac{1}{4}(2S_i + S_i + S_k)$$

(Basic properties)

We state some basic properties on Mendel algebras:

(1) Hardy-Weinberg law([4]):

We have the following equality:

$$(\sum_{i=1}^{n} q_i S_i)^2 = \sum_{i=1}^{n} q_i S_i \ (\sum_{i=1}^{n} q_i = 1)$$

(2)Mendel algebra with defects([4]):

In order to treat mutations of genetics in the realistic manner, we have to introduce a concept of defects in Mendelian genetics. It is called of homo type, when $S_i * S_j = 0\ (^{\exists}i, ^{\exists}j, i \neq j)$. It is called of hetero type, when $S_i^2 = 0\ (^{\exists}i)$.

(3)Gametic algebra([5])

We can generalize a concept of Mendel algebra to gametic algebra. Let $R_m[x_0,x_1,..,x_n]$ be the space of homogenous polynomials with n+1 variables of degree m. Choosing $f,g \in R_m[x_0,x_1,..,x_n]$ and define the product of f and g by

$$f * g = \frac{m!}{2m!}\frac{\partial^m}{\partial x_0{}^m}(fg),$$

we obtain the gametic algebra of degree m. In the case of degree 1, it becomes the Mendel algebra([5]).

(4)Tensor product of Mendel algebras

We can define the tensor product $M_1 \otimes M_2$ of two Mendel algebras M_1, M_2 as follows:

Putting $M_1 = R[S_1,S_2,..,S_n]$, $M_2 = R[S'_1, S'_2,..,S'_m]$, we define

$$M_1 \otimes M_2 = R[S_i \otimes S'_j : i = 1,2,..,n, j = 1,2,..,m],$$

where we define the product by

$$(S_i \otimes S'_j)\hat{*}(S_k \otimes S'_l) = (S_i * S_k) \otimes (S'_j *' S'_l).$$

Then we have the following proposition:

Proposition(Independent law)
We have the following separation law:

(1) $(S_i \otimes S'_j)\hat{*}(S_k \otimes S'_l) = 1/2^2(S_i \otimes S'_j + S_i \otimes S'_l + S_k \otimes S'_j + S_k \otimes S'_l)$

(2)Putting $X = \sum_{i=1}^{n} \alpha_i S_i, Y = \sum_{j=1}^{m} \beta_j S'_j$ and $U = \sum_{i=1}^{n} \alpha'_i S_i, V = \sum_{j=1}^{m} \beta'_j S'_j$, we have $X \otimes Y = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j S_i \otimes S'_j$, $U \otimes V = \sum_{k=1}^{n} \sum_{l=1}^{m} \alpha'_k \beta'_l S_k \otimes S'_l$. Then we have

$(X \otimes Y) * (U \otimes V) = \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha'_k \beta_j \beta'_l (S_i \otimes S'_j + S_i \otimes S'_l + S_k \otimes S'_j + S_k \otimes S')$

Remark. We have considered the vector space $M^{(n)}(R)$ over R. From biological point of view, it seems to be not natural. In this paper we consider mathemtics which is motivated by genetics. If you want to treat it in a biological manner regolously, we may consider the vector space over finite field F, for example, $F = \{0,1,2,..,N\}$ where N is the total set of life things, or DNA.

3. Mendel algebra is flexible algebra

In this section we treat flexible algebras from our point of view. We begin with the definition([6]):

Definition
(1)An algebra A is called flexible algebra, if the following commutation relation is satisfied:

$$\forall X, \forall Y \in A \Rightarrow (XY)X = X(YX)$$

At first we give a simple flexible algebra. We choose an algebra with the following product table:

	e_1	e_2
e_1	$2e_1$	e_2
e_2	e_2	0

$\Longrightarrow \quad (e_1 e_1)e_2 \neq e_1(e_1 e_2)$

Proposition
The algebra is a flexible algebra, but not an associative algebra.

Proof
Putting $X = \sum \alpha_i e_i$, $Y = \sum \beta_i e_i$, we check the condition: $(XY)X = X(YX)$. Since $XY = YX = 2x_1 y_1 e_1 + (x_1 y_2 + x_2 y_1)e_2$. Hence we have $(XY)X = 2x_1^2 y_1 e_1 + x_1 x_2 y_1 e_2$ and $(X(YX) = 2x_1^2 y_1 e_1 + x_1 x_2 y_1 e_2$ which proves the assertion.

Next we proceed to flexible algebras generated by Mendel algebras.

Theorem I
Mendel algebra $M^{(n)}$ $(n \geq 2)$ is a flexible algebra, but not an associative algebra.

Proof Putting $X=\sum \alpha_i S_i, Y=\sum \beta_i S_i$, we see
$((XY)X)=\sum \alpha_i \beta_j \alpha_k (S_i * S_j) * S_k$, and $(X(YX))=\sum \alpha_i \beta_j \alpha_k S_i * (S_j * S_k)$.

Hence to prove the assertion, it is enough to prove the following equality:

$$\sum \alpha_i \beta_j \alpha_k (S_i * S_j) * S_k = \sum \alpha_i \beta_j \alpha_k S_i * (S_j * S_k).$$

For this we decompose the both sides in the following manner:

$$\sum \alpha_i \beta_j \alpha_k (S_i * S_j) * S_k = \sum_{i=k} \alpha_i \beta_j \alpha_k (S_i * S_j) * S_k + \sum_{i\neq k} \alpha_i \beta_j \alpha_k (S_i * S_j) * S_k$$
$$\sum \alpha_i \beta_j \alpha_k S_i * (S_j * S_k) = \sum_{i=k} \alpha_i \beta_j \alpha_k S_i * (S_j * S_k) + \sum_{i\neq k} \alpha_i \beta_j \alpha_k S_i * (S_j * S_k)$$

Since $((S_i * S_j) * S_i) = ((S_i * (S_j * S_i))$, the first term of the both sides are identical. The second terms of the both sides can be written as follows:

$$\sum_{i\neq k} \alpha_i \beta_j \alpha_k (S_i * S_j) * S_k = \sum_{i\neq k} \alpha_i \beta_j \alpha_k \{(S_i * S_j) * S_k + (S_k * S_j) * S_i\}$$
$$\sum_{i\neq k} \alpha_i \beta_j \alpha_k S_i * (S_j * S_k) = \sum_{i\neq k} \alpha_i \beta_j \alpha_k \{S_i * (S_j * S_k) + S_k * (S_j * S_i)\}$$

By use of the commutativity of Mendel algebra, we see the both sides are identical. Hence we have proved the assertion.

4. Mendel algebra is Jordan algebra

In this section we make a Jordan algebra from our point of view. We begin with the definition([3],[6]):

Definition
An algebra J is called Jordan algebra if the following commutation relation holds for $\forall X, \forall Y \in J$:

$$(((XX)Y)X) = ((XX)(YX)).$$

When it is commutative, it is called Jordan algebra simply. Otherwise it is called non-commutative Jordan algebra.

At first we give a simple non-commutative Jordan algebra.

Proposition
The algebra with the following product table is a (non-commutative) Jordan algebra

	e_1	e_2
e_1	e_1	e_2
e_2	$-e_2$	e_1

$\Longrightarrow \quad (e_2 e_1) e_2 \neq e_2 (e_1 e_2)$

Proof
The proof is a direct calculation. Putting $X=\sum x_i e_i, Y=\sum y_j e_j$, we have

$$(XX)=(x_1^2+x_2^2)e_1, \text{and} (YX)=(x_1 y_1 + x_2 y_2)e_1 + (x_2 y_1 + x_1 y_2)e_2$$

From $(XX)Y=(x_1^2+x_2^2)(y_1 e_1 + y_2 e_2)$, we have

$$((XX)Y)X=(x_1^2+x_2^2)\{(x_1 y_1 + x_2 y_2)e_1 + (x_2 y_1 - x_1 y_2)e_2)\}.$$

On the other side we have

$$((XX)(YX)) = (x_1^2 + x_2^2)\{(x_1y_1 + x_2y_2)e_1 + (x_2y_1 - x_1y_2)e_2)\}.$$

Hence we have the assertion.

Next we proceed to the realization of Jordan algebra by use of the Mendelian algebra $M^{(n)}$. We can prove the following:

Theorem II

Mendel algebra $M^{(n)}$ $(n \geq 2)$ is a Jordan algebra,but not an associative algebra .

Proof : Putting $X = \sum \alpha_i S_i, Y = \sum \beta_i S_i$, we have

$(((XX)Y)X) = \sum \alpha_i \alpha_j \beta_k \alpha_l ((S_i * S_j) * S_k) * S_l$,

$((XX)(YX)) = \sum \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l)$,

Hence to prove the assertion, it is enough to prove the following equality:

$$\sum \alpha_i \alpha_j \beta_k \alpha_l ((S_i * S_j) * S_k) * S_l = \sum \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l) \cdot$$

For this we decompose the both sides in the following manner:

$$\sum \alpha_i \alpha_j \beta_k \alpha_l ((S_i * S_j) * S_k) * S_l = \sum_{i=j=l} \alpha_i \beta_j \alpha_k (S_i * S_j) * S_k + \sum{}' \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l)$$
$$\sum \alpha_i \alpha_j \beta_k \alpha_k (S_i * (S_j * S_k)) * S_l = \sum_{i=k=l} \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l) + \sum{}' \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l),$$

where the second sum is remained sum. Since $((S_i * S_i) * S_k) * S_i = ((S_i * S_i) * (S_k * S_i))$,

the first term of the both sides are identical. Next we decompose the remained sum into two parts: $\Sigma' = \Sigma'_1 + \Sigma'_2$:The first sum is taken for the case of two of the indices (i, j, l) are identical and the remained sum is taken for three different indices . The second terms of the both sides can be written as follows:

$$\sum{}_2' \alpha_i \alpha_j \beta_k \alpha_l ((S_i * S_j) * S_k) * S_l = \sum_\sigma \alpha_{\sigma(i)} \alpha_{\sigma(j)} \beta_k \alpha_{\sigma(l)} \{(S_{\sigma(i)} * S_{\sigma(j)}) * S_k * S_{\sigma(l)}\}$$
$$\sum{}_2' \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l) = \sum_\sigma \alpha_{\sigma(i)} \alpha_{\sigma(j)} \beta_k \alpha_{\sigma(l)} \{(S_{\sigma(i)} * S_{\sigma(j)}) * (S_k * S_{\sigma(l)})\},$$

where the sum is taken through the permutations of three words. By use of

$$(****)(((S_i * S_j) * S_k) * S_l) = \frac{1}{8}(S_i + S_j + 2S_k + 4S_l), ((S_i * S_j) * (S_k * S_l)) = \frac{1}{4}(S_i + S_j + S_k + S_l),$$

we see the botth sides are identical. In a completely analogous manner, we have the first equality for Σ'_1 . Hence we have proved the assertion.

5. Genetic generations of non-associative algebras

In this section we introduce genetic constructions of non-associative algebras from a given non-associative algebra following Mendel's laws in Section 1. By this method, we can generate a class of non-associative algebras including flexible and Jordan algebras systematically.

(1) Separation law

At first we notice that we can introduce a Mendel algebra $M(A)$ from a given (associative) algebra A which is finitely generated: $A = R[e_1, e_2, .., e_n]$. We see that $x \in A$ has the following form: $x = \sum \alpha_{i_1 i_2 .. i_n} \Omega_{i_1 i_2 .. i_n}$ $(\Omega_{i_1 i_2 .. i_n} = e_1^{i_1} e_2^{i_2} ... e_n^{i_n})$. Then we can define the Mendel algebra by the following product rule with a distributive law:

$$\Omega_{i_1 i_2 \dots i_n} * \Omega_{i_1 i_2 \dots i_n} = \frac{1}{2}(\Omega_{i_1 i_2 \dots i_n} + \Omega_{i_1 i_2 \dots i_n})$$

Then we can prove the following proposition:

Proposition
Mendel algebra $M(A)$ of an algebra A is a Mendel algebra which is generated by the linear basis of A.

(2)Mating law
We make the following definition:

Definition
For an algebra A, putting

$$X \circ Y = \frac{1}{2}(XY + YX) \text{ for } X, Y \in A,$$

we have an algebra which is called the specialization algebra of A

Then we can prove the following Proposition:

Proposition
(1)The specialization of Mendel algebra is the original Mendel algebra.
(2) The specialization of a flexible algebra is a flexible algebra.
(3) The specialization of a commutative Jordan algebra is a Jordan algebra..
Proof of (1). If follows from the definition.

Proof of (2). Let A be a flexible algebra. For elements $X, Y \in A$, we have

$$(X \circ Y) \circ X = \frac{1}{4}\{(XY)X + (YX)X + X(XY) + X(YX)\}$$

$$X \circ (Y \circ X) = \frac{1}{4}\{(X(XY) + X(YX) + (YX)X + X(YX)\}$$

Hence comparing the both sides and using the commutation relation, we have the assertion.

Proof of (3). $(((X \circ X) \circ Y) \circ X)$

$$= \frac{1}{4}\{((Y(XX))X + X((Y(XX))\} + \frac{1}{4}\{((XX)Y)X + X((XX)Y))\}$$

$$((X \circ X) \circ (Y \circ X))$$

$$= \frac{1}{4}\{(XX)(YX) + (YX)(XX)\} + \frac{1}{4}\{(XX)(XY) + (XY)(XX)\}$$

By use of the commutativity and the commutation relation, we have the assertion.

(3)Independent law
We make the following definition:

Definition
For algebras A, A', putting $X \otimes Y'$ for $X \in A, Y' \in A'$, we have an algebra which is called the tensor product of A and A'.

Then we can prove the following Proposition:

Proposition
(1)The tensor product of Mendel algebras is a Mendel algebra.
(2)The tensor product of flexible algebras is a flexible algebra.
(3)The tensor product of Jordan algebras is a Jordan algebra.

Proof of (1). It follows from the definition.
Proof of (2). Let A, A' be flexible algebras. For $\hat{X} = X \otimes X', \hat{Y} = Y \otimes Y'$, we define $\hat{X} \bullet \hat{Y} = (X \otimes X') \bullet (Y \otimes Y') = (XY) \otimes (X'Y')$. Then we have

$$(\hat{X} \bullet \hat{Y}) \bullet \hat{X} = \{(X \otimes X') \bullet (Y \otimes Y')\} \bullet \hat{X} = \{(XY) \otimes (X'Y')\} \bullet X \otimes X' = (XY)X \otimes (X'Y')X'$$

$$\hat{X} \bullet (\hat{Y} \bullet \hat{X}) = X \bullet \{(Y \otimes Y') \bullet (X \otimes X')\} = X \otimes X \bullet \{(XY) \otimes (X'Y')\} = X(YX) \otimes (X'(Y'X')$$

Hence we have the assertion.
Proof of (3). Let A, A' be Jordan algebras. For $\hat{X} = X \otimes X', \hat{Y} = Y \otimes Y'$, we define $\hat{X} \bullet \hat{Y} = (X \otimes X') \bullet (Y \otimes Y') = (XY) \otimes (X'Y')$. Then we have

$$\begin{aligned}((\hat{X} \bullet \hat{X}) \bullet \hat{Y}) \bullet \hat{X} &= \{\{(X \otimes X') \bullet (X \otimes X')\} \bullet (Y \otimes Y')\}\} \bullet X \otimes X' \\ &= \{\{(XX) \otimes (X'X')\} \bullet Y \otimes Y'\} \bullet X \otimes X = \{(XX)Y \otimes (X'X')Y'\} \bullet \{X \otimes X\} \\ &= ((XX)Y)X \otimes ((X'X')Y')X\end{aligned}$$

$$\begin{aligned}(\hat{X} \bullet \hat{X}) \bullet (\hat{Y} \bullet \hat{X}) &= \{(X \otimes X') \bullet (X \otimes X')\} \bullet \{(Y \otimes Y') \bullet (X \otimes X')\} \\ &= \{(XX) \otimes (X'X')\} \bullet \{(YX) \otimes (Y'X')\} = \{(XX)(YX) \otimes (X'X')(Y'X')\}\end{aligned}$$

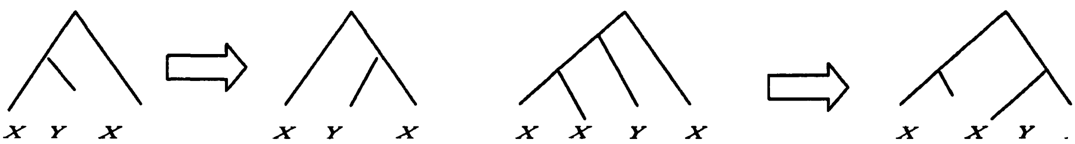
Hence we have the assertion.

Summarizing the results above mentioned, we have the following Theorem:

Theorem III
When we call the three processes above mensioned in one word genetic generation method, we can prove the following results:
(1)We can generate Mendel algebras by the genetic generation method .
(2) Flexible algebra is generated by the genetic generation method.
(3) Jordan algebra is generated by the genetic generation method.

6. Classifications of non-associative algebras based on Mendel algebras

In this section we shall obtain flexible algebra and Jordan algebra from the shift invariant conditions. Here shift implies that the change of the neighboring brackets in an acceptable manner in the sense of formal language and shift invariance implies the elements give the same elements by the shifts of brackets. We give two examples of shift operations.



flexible algbera Jordan algbera

Based on this fact, we can get a group of non-associative algebras which are related to Mendel algebras.

Proposition(Shift invariance of flexible algebra)
We assume the following shift invariant elements: $X*(Y*Z) = (X*Y)*Z$ for $\forall X, \forall Y, \forall Z \in M(A)$. Then we have $X* = Z*$. Hence we have a flexible algebra.

Proof Putting $X = \sum \alpha_i S_i, Y = \sum \beta_i S_i, Z = \sum \gamma_i S_i$ we consider the shift invariant condition: $X*(Y*Z) = (X*Y)*Z$. Restricting special element, we consider $((S_i * S_j) * S_k) = ((S_i * (S_j * S_k))$. Then we see $S_i = S_k$. we have

$((XY)X)=\sum\alpha_i\beta_j\alpha_k\delta_{ik}(S_i*S_j)*S_k$, and $(X(YX))=\sum\alpha_i\beta_j\alpha_k\delta_{ik}S_i*(S_j*S_k)$, Hence we obtain $X*(Y*X)=(X*Y)*X$

Proposition(Shift invariance of Jordan algebra)
We assume that $((X*Y)*Z)*W=(X*Y)*(Z*W)$. Then we have $X=Y=W$. Hence we have a Jordan algebra.

Proof : From

$$(****)(((S_i*S_j)*S_k)*S_l)=\frac{1}{8}(S_i+S_j+2S_k+4S_l),((S_i*S_j)*(S_k*S_l))=\frac{1}{4}(S_i+S_j+S_k+S_l)$$

and $(((S_i*S_j)*S_k)*S_l)=((S_i*S_j)*(S_k*S_l))$, we have $S_i=S_l=S_l$. Hence putting $X=\sum\alpha_iS_i, Y=\sum\beta_iS_i$, we have the commutation relation of a Jordan algebra.

Hence we see that the shift invariance condition chooses a class of non-associative algebras in Mendel algebras. Therefore we may expect to list up non-associative algebras connected to Mendel algebras using the shift invariance of elements in the following table:

(The table of possible commutation relations)
(1)The terms of shift invariant conditions of degree 3

$((XY)Z), (X(YZ))$

(2) The terms of shift invariant conditions of degree 4

$(((XY)Z)W),\ ((X(YZ))W),\ ((XY)(ZW)),\ (X((Y(ZW)),\ (X((YZ)W))$

(3) The terms of shift invariant conditions of degree 5

$(((XY)Z)W)U,\ (X(YZ))W)U,\ (X((YZ)W))U,\ (X(Y(ZW)))U,\ (X(Y(Z(WU))),\ X(Y((ZW)U))$
$X((Y(ZW))U),\ X(((YZ))W)U),\ ((XY)(ZW))U,\ ((XY)Z)(WU),\ (XY)(Z(WU)),\ X((YZ)(WU))$

Examples of calculations of shift invariant elements tell us that the commutation relations of flexible algebra and Jordan algebra are basic and that we can get the algebras with commutation relations which are generated by those of flexible algebras and Jordan algebras.

References

[1] Y.Asou,H.Kouriyama, M.Matsuo, K.Nouno and O.Suzuki Construction of non-associative algebras in algebras generated by Chomsky sentences (To appear in the RIMS Proceeding Lecture Note

[2] N. Chomsky:Context-free grammar and pushdown storge,Quaterly Prog. Rep. No.65,187-194,CambridgeMass(1952)

[3]N.Jackobson:General representation theory of Jordan algebras,Trans. Amer. Math. Soc. 70(1951)

[4] D.L.Hartl: Essential genetics:A genomics perspective,John and Bartlett Publishers, Inc.M.A.U.S.A(2002)p.519

[5] A.Micali and Ph. Revoy:Sur les algebre gametique,Proc. Edinburgh Math. Soc. 29(1986),197-206

[6] T.A.Springer and F.D.Veldkamp:Octonions, Jordan algebra and exceptional group, Springer Monograph in Mathematics, Springer 2000